MATHEMATICS

NOTES ON BANACH FUNCTION SPACES, XIVA

BY

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The present note is a continuation of earlier notes under the same title [6]. We start with a discussion of separable spaces which are σ-Dedekind complete (and hence super Dedekind complete by Corollary 42.4 of Note XIII). One of the main results here is that if L_ϱ is separable and σ-Dedekind complete, then every monotone orderbounded sequence in L_ϱ is a ϱ-Cauchy sequence. We also investigate the connection between the Egoroff property and the ideal L_ϱ^a; it is shown that in any separable L_ϱ the ideal L_ϱ^a is order dense if and only if L_ϱ has the Egoroff property. After that we present simple proofs of some reflexivity theorems for separable normed spaces. A separate section is devoted to a discussion of separable normed function spaces. The main result here is that if ϱ is a function norm and L_ϱ is separable, then L_ϱ^a is an order dense ideal. Furthermore it is pointed out that separability of L_ϱ implies separability of the underlying measure μ, and separability of μ implies separability of L_ϱ^a.

Improving upon Theorem 35.9 in Note XI it has been shown by Professor T. Andô that if L_ϱ is a normed Riesz space with the property that every norm closed ideal is a normal subspace, then ϱ is a normal norm (i.e., $u_\tau \downarrow 0$ in L_ϱ implies $\varrho(u_\tau) \downarrow 0$). This has some interesting additional consequences. We hereby thank Professor T. Andô for his permission to include this theorem and its proof in the present note.

43. *Separability and ordercompleteness*

Theorem 42.8 of Note XIII yields a satisfactory result for separable spaces L_ϱ which are norm complete, or which have at least the property that every ϱ-Cauchy sequence $\{u_n\}$ with $u_n \downarrow 0$ satisfies $\varrho(u_n) \downarrow 0$. It is of interest to investigate also separable spaces L_ϱ which do not have this property. The following theorem, in a certain sense, parallels Lemma 42.6 of Note XIII.

Theorem 43.1. *If L_ϱ is separable and σ-Dedekind complete, then ϱ has* (A, iii), *i.e., every monotone orderbounded sequence in L_ϱ is a ϱ-Cauchy sequence.*

Proof. If the theorem is false, then there exists a sequence $0 \leqslant u_k \uparrow \leqslant u$ and a number $\varepsilon > 0$ such that $\varrho(u_{k+1} - u_k) > \varepsilon$ for $k = 1, 2, \ldots$, and so there

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exist also elements $0 \leqslant \varphi_k \in L_\varrho^*$ satisfying $\varrho^*(\varphi_k)=1$ and $\varphi_k(u_{k+1}-u_k)>\varepsilon$ for all k. According to the remarks made in section 42 of Note XIII, the norm closed unit ball S^* of L_ϱ^* is a compact metric space in the $\sigma(S^*, L_\varrho)$ topology, and hence there exists a subsequence $\varphi_{k_n}(n=1, 2, \ldots)$ such that $\varphi_{k_n}(f)$ converges as $n \to \infty$ for every $f \in L_\varrho$. For brevity we set $\varphi_{k_n}=\psi_n$ and $u_{k_n+1}-u_{k_n}=v_n$. For every subset A of the set N of all natural numbers the element $v_A=\sum_{n \in A} v_n$ exists by the σ-Dedekind completeness of L_ϱ, and now we set $\mu_n(A)=\psi_n(v_A)$ for every such set A. Then for every n, μ_n is a nonnegative and finitely additive measure on the Boolean algebra of all subsets of N, and $\lim \mu_n(A)=\mu(A)$ exists for every $A \subset N$. Obviously, μ is again a non-negative and finitely additive measure. It follows that $\nu_n=\mu_n-\mu$ is finitely additive and $\lim \nu_n(A)=0$ as $n \to \infty$ for every subset $A \subset N$. According to a result due to R. S. PHILLIPS ([9], Lemma 3.3) this implies that $\lim \sum_{k=1}^{\infty} |\nu_n(\{k\})|=0$ as $n \to \infty$, and so surely $\lim \nu_n(\{n\})=0$ as $n \to \infty$. Since $\mu(\{n\}) \to 0$ as $n \to \infty$ it follows that $\mu_n(\{n\}) \to 0$, i.e., $\varphi_{k_n}(u_{k_n+1}-u_{k_n}) \to 0$ as $n \to \infty$. Contradiction.

For the proof of the following theorem we insert first the following basic lemma which generalizes Lemma 31.1 in Note IX.

Lemma 43.2. *Let ϱ be a Riesz seminorm on the Riesz space L. If $0 \leqslant u_n \leqslant u \in L$ for $n=1, 2, \ldots$ and $\sum \varrho(u_n)<\infty$, then there exist for every $\varepsilon>0$ and for every Riesz seminorm λ having the property that every monotone orderbounded sequence is a λ-Cauchy sequence, i.e.,* (A, iii), *sequences $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ of positive elements in L such that* (a) *$0 \leqslant v_n \leqslant z_n$ and $\varrho(z_n) \downarrow 0$,* (b) *$\lambda(w_n)<\varepsilon$ for all $n=1, 2, \ldots$ and* (c) *$u_n=v_n+w_n$ for all n.*

Proof. Let $\varepsilon>0$ and let λ be a Riesz seminorm having the property (A, iii). Assume that $\sum \varrho(u_n)<\infty$ and $0 \leqslant u_n \leqslant u$ for all n. For every n, set $u_{n,m}=\sup(u_n, \ldots, u_m)$ for $m \geqslant n$. Then $u_n \leqslant u_{n,m} \uparrow_m \leqslant u$, and so $u_{n,m}(m \geqslant n)$ is a λ-Cauchy sequence in m for every n. Hence, there exists an index sequence m_n, increasing as n increases, such that, for $n=1, 2, \ldots$,

$$\lambda(u_{n,k}-u_{n,m_n})<\varepsilon/2^n \text{ for all } k \geqslant m_n.$$

For brevity, set $x_n=u_{n,m_n}$ and $z_n=\inf(x_1, \ldots, x_n)$. Then $z_n \downarrow$, and it follows from $z_n \leqslant x_n=u_{n,m_n} \leqslant u_n+\ldots+u_{m_n}$ that $\varrho(z_n) \downarrow 0$ as $n \to \infty$.

Now, for every n, let

$$y_n=\sum_{i=1}^{n-1} [\sup(x_i, \ldots, x_n)-x_i]=\sum_{i=1}^{n-1} (u_{i,m_n}-u_{i,m_i}).$$

Then $\lambda(y_n)<\sum_1^{\infty} \varepsilon/2^i=\varepsilon$ for all n. Furthermore, $x_n-y_n \leqslant x_i$ for $i=1, 2, \ldots, n$ and so $x_n-y_n \leqslant z_n$ for $n=1, 2, \ldots$. Since $u_n \leqslant x_n$, this implies that $u_n \leqslant y_n+z_n$ for all n. Finally, let $v_n=\inf(u_n, z_n)$ and $w_n=u_n-v_n$ for all n. Then $0 \leqslant v_n \leqslant z_n$ and $\varrho(z_n) \downarrow 0$, and $u_n=v_n+w_n$, so the conditions (a) and (c) are satisfied. For (b), observe that

$$w_n = u_n - v_n = \inf (y_n + z_n, u_n) - \inf (z_n, u_n) \leqslant y_n + z_n - z_n = y_n,$$

so $\lambda(w_n) \leqslant \lambda(y_n) < \varepsilon$ for all n.

For separable and σ-Dedekind complete L_ϱ spaces, Theorem 42.3 in Note XIII can be strengthened as follows.

Theorem 43.3. *If L_ϱ is separable and σ-Dedekind complete, then there exists a strictly positive linear functional $\varphi \in L_\varrho^*$ with the following properties.*

(i) $0 \leqslant u_\tau \downarrow$ *and* $\varphi(u_\tau) \downarrow 0$ *implies* $\varrho(u_\tau) \downarrow 0$.

(ii) $0 \leqslant u_n \leqslant u (n = 1, 2, \ldots)$ *and* $\lim \varphi(u_n) = 0$ *as* $n \to \infty$ *implies* $\lim \varrho(u_n) = 0$ *as* $n \to \infty$.

Proof. (i) Since the norm completion $\bar{L}_\varrho$ of L_ϱ is also separable whenever L_ϱ is separable it follows from Theorem 42.3 in Note XIII that there exists an element $\varphi \in L_\varrho^*$ which is strictly positive on the norm completion $\bar{L}_\varrho$ of L_ϱ. Let $0 \leqslant u_\tau \downarrow$, $u_\tau \in L_\varrho$ for all τ, and let $\varphi(u_\tau) \downarrow 0$. Then $u_\tau \downarrow 0$ in $\bar{L}_\varrho$. From the hypotheses that L_ϱ is separable and σ-Dedekind complete it follows by means of Theorem 43.1 that $\{u_\tau\}$ is a ϱ-Cauchy net. Thus $\{u_\tau\}$ converges in norm to $0 = \inf u_\tau$ in $\bar{L}_\varrho$, i.e., $\varrho(u_\tau) \downarrow 0$.

(ii) Let $0 \leqslant u_n \leqslant u$ for all n and let $\varphi(u_n) \to 0$ as $n \to \infty$. In order to show that $\varrho(u_n) \to 0$ as $n \to \infty$ there is no loss in generality to assume that $\sum \varphi(u_n) < \infty$. Since by Theorem 43.1 ϱ has (A, iii) it follows from Lemma 43.2 that for every $\varepsilon > 0$ there exist sequences $\{v_n\}$, $\{w_n\}$, $\{z_n\}$ of positive elements such that (a) $v_n \leqslant z_n \downarrow$ and $\varphi(z_n) \downarrow 0$, (b) $\varrho(w_n) \leqslant \varepsilon$ for all n and (c) $u_n = v_n + w_n$ for all n. By (i), $\varphi(z_n) \downarrow 0$ implies $\varrho(z_n) \downarrow 0$, and hence $\limsup \varrho(u_n) \leqslant \varepsilon$. Thus $\varrho(u_n) \to 0$ as $n \to \infty$.

We are now able to prove the following interesting theorem.

Theorem 43.4. *If L_ϱ is separable and σ-Dedekind complete, then ϱ has the following property* (A, v). *For every $0 < u \in L_\varrho$ and for every $\varepsilon > 0$ there exists a natural number $n_0 = n_0(u, \varepsilon)$ such that if $u \geqslant \sum_{k=1}^{n} u_k$ and $\varrho(u_k) \geqslant \varepsilon$, then $n \leqslant n_0$.*

Proof. If ϱ does not have the property (A, v), then there exist an element $0 < u \in L_\varrho$, a positive number $\varepsilon_0 > 0$ and elements $u_{i,j} (i = 1, 2, \ldots, j = 1, 2, \ldots, n_i)$ such that $u \geqslant \sum_{j=1}^{n_i} u_{i,j} (i = 1, 2, \ldots)$, $\varrho(u_{i,j}) \geqslant \varepsilon_0 > 0$ and $n_i \geqslant 2^i$ $(i = 1, 2, \ldots)$. Let $0 \leqslant \varphi \in L_\varrho^*$ be a strictly positive linear functional on L_ϱ having the properties (i) and (ii) of Theorem 43.3. There is no loss in generality to assume that $\varphi(u) = 1$. Then for every i there exists an index $j_i (i = 1, 2, \ldots)$ such that $\varphi(u_{i,j_i}) \leqslant 2^{-i}$. For brevity we set $u_i = u_{i,j_i}$ $(i = 1, 2, \ldots)$. Then $0 \leqslant u_i \leqslant u$ and $\sum \varphi(u_i) < \infty$. Hence, by (ii) of Theorem 43.3, $\varrho(u_i) \to 0$ as $i \to \infty$, contradicting $\varrho(u_i) = \varrho(u_{i,j_i}) \geqslant \varepsilon_0 > 0$.

It is easy to see that (A, v) implies (A, iii). It will be shown later that in fact (A, iii) and (A, v) are equivalent.

There exist separable and norm complete spaces L_ϱ which do not have the property that every monotone orderbounded sequence is a ϱ-Cauchy sequence and hence such spaces are not σ-Dedekind complete (Example: the space of all real continuous functions on [0, 1]). It will also be shown later that L_ϱ has the property (A, iii) if and only if the norm completion of L_ϱ has the same property. But it is obvious that a norm complete space L_ϱ having the property (A, iii) has the stronger property (A, ii) ($u_\tau \downarrow 0$ implies $\varrho(u_\tau) \downarrow 0$). Hence, the norm completion of a separable and σ-Dedekind complete space L_ϱ is separable, super Dedekind complete, and (A, ii) holds. It follows in particular that the space of all real continuous functions on [0, 1] cannot possess any norm dense Riesz subspace which is σ-Dedekind complete.

It is natural to ask whether the norm ϱ of a separable and σ-Dedekind complete space L_ϱ satisfies the stronger condition (A, ii) ($u_\tau \downarrow 0$ implies $\varrho(u_\tau) \downarrow 0$) or at least the property (A, ii) on some order dense ideal in L_ϱ. That this is not the case will be shown by means of the following example.

Example 43.5. Let $\mathfrak{B}$ denote the Boolean algebra of all regular open subsets of the open interval (0, 1) of the real line (A subset of a topological space is called regular open if the interior of the closure coincides with the set). It is well-known that $\mathfrak{B}$ is a complete Boolean algebra ([1], Theorem 6 of Chapter XI). Let X be the Stone representation space of $\mathfrak{B}$. Then X is a zero dimensional compact Hausdorff space. Since $\mathfrak{B}$ is complete X is extremally disconnected, i.e., the closure of every open set is open ([2], Theorem 11.3). The Boolean algebra $\mathfrak{B}$ has obviously no atoms, i.e., for every $A \in \mathfrak{B}$ there is an element $A_1 \in \mathfrak{B}$ such that $A_1 \subset A$ and $A_1 \neq A$. Hence, X has no isolated points. Finally, it will be shown that X has a countable dense subset, i.e., X is separable. To this end we recall that X consists of the set of all maximal filters of $\mathfrak{B}$. Let $\{r_n\}$ be the set of rationals in (0, 1). To every rational r_n there corresponds the unique maximal filter $x_n \in X$ of $\mathfrak{B}$ consisting of all $A \in \mathfrak{B}$ such that $r_n \in \bar{A}$, the closure of A. Then, the set $\{x_n\}$ is dense in X. Indeed, since the open and closed subsets of X form a basis for the topology of X it follows that it is sufficient to show that every open and closed subset of X contains at least one element of $\{x_n\}$ which is evident, however, from the fact that every regular open set contains at least one rational and the fact that every open and closed subset of X consists of all maximal filters which contain a given element of $\mathfrak{B}$. Thus, in summary, X is a zero dimensional separable compact Hausdorff space without isolated points such that the closure of every open set is open.

Let L be the Riesz space of all real continuous functions on X. Since the closure of every open set is open it follows from a theorem of M. H.

STONE [11] and H. NAKANO [1] that L is Dedekind complete. Furthermore, the separability of X implies that L admits a strictly positive linear functional. Indeed, if $\{x_n\}$ is dense in X, then $\varphi_0(f)=\sum_1^\infty f(x_n)/2^n$, $f\in L$, is a strictly positive linear functional on L. Hence, by Lemma 27.16 in Note VIII, L is super Dedekind complete, and so in particular, every integral on L is normal. The next additional property of L we shall prove is that L does not admit any normal integral other than the zero functional. To this end, we observe that for every $x_n (n=1, 2, \ldots)$ there exists a sequence $1\geqslant u_{n,m}\geqslant 0$, $\downarrow m$ in L such that $u_{n,m}(x_n)=1$ for all n and m and for every n, $\inf_m u_{n,m}=0$ in L. If $0\leqslant\varphi\in L_n^\sim$ and $\varepsilon>0$ are given, then for every n there exists an index m_n such that $\varphi(u_{n,m_n})\leqslant\varepsilon/2^n$. Since the set $\{x_n\}$ is dense in X and $u_{n,m_n}(x_n)=1$ and $u_{n,m_n}\leqslant 1$ for all n it follows that $\sup_n u_{n,m_n}=1$ in L. Hence, $\varphi(1)\leqslant\sum_1^\infty\varphi(u_{n,m_n})\leqslant\varepsilon$, i.e., $\varphi(f)=0$ for all $f\in L$.

Consider now the following norm $\varrho(f)=\sum_1^\infty|f(x_n)|/2^n$ on L, where $\{x_n\}$ is a countable dense subset of X. Observing that for every $f\in L$, $\{f(x_n)\}$ is a bounded sequence of real numbers such that if $f, g\in L$, then $f=g$ if and only if $f(x_n)=g(x_n)$ for all n it follows immediately that L_ϱ is isomorphic and isometric with a Riesz subspace of the Riesz space of all bounded real sequences with the same norm. Since the latter space is separable it follows that L_ϱ is separable also. Finally, $L_n^\sim=\{0\}$ implies that $L_{\varrho,c}^*=L_{\varrho,n}^*=\{0\}$, so $L_{\varrho,s}^*=L_\varrho^*$, and hence $L_\varrho^a={}^\perp(L_{\varrho,s}^*)=\{0\}$. Thus L_ϱ is a separable super Dedekind complete Riesz space such that $L_\varrho^a=\{0\}$. From Corollary 20.7 of Note VI it follows that L does not have the Egoroff property. Furthermore, $L=C(X)$ in the uniform norm is not separable.

44. *Separability and the Egoroff property*

In this section we shall derive necessary and sufficient conditions in order that, in a separable space L_ϱ, the ideal L_ϱ^a is order dense. Since in a separable L_ϱ the restriction of the norm ϱ to L_ϱ^a is normal (i.e., $u_\tau\downarrow 0$ in L_ϱ^a implies $\varrho(u_\tau)\downarrow 0$), it follows from Theorem 35.1 in Note XI that L_ϱ^a has the Egoroff property, and hence, if L_ϱ^a is order dense, the Egoroff property holds already on an order dense ideal. It will be shown now that, in a separable L_ϱ, the ideal L_ϱ^a is order dense if and only if the Egoroff property holds in the whole space L_ϱ.

If A is an ideal in the arbitrary Riesz space L, we shall say that A is *super order dense* if, given any $u\geqslant 0$ in L, there exists a sequence $0\leqslant u_n\uparrow u$ with $u_n\in A$ for all n. Obviously, any super order dense ideal is order dense, and in a separable L_ϱ any order dense ideal is super order dense.

Lemma 44.1. *Let L be a Riesz space having the Egoroff property. If $\{A_n: n=1, 2, \ldots\}$ is a sequence of super order dense ideals in L, then the ideal $\bigcap_{n=1}^\infty A_n$ is super order dense.*

Proof. Given $0<u \in L$, there exists, for every n, a sequence $0 \leqslant u_{nk} \uparrow_k u$ such that $u_{nk} \in A_n$ for all k. Hence, by the Egoroff property, there exists a sequence $0 \leqslant u_m \uparrow u$ with the property that for every pair (m, n) there is an index $j(m, n)$ such that $u_m \leqslant u_{n,j(m,n)}$. Hence $u_m \in A_n$ for all n, i.e., $u_m \in \cap A_n$. Since this holds for all m, it follows that $\cap A_n$ is super order dense.

Theorem 44.2. *Let L_ϱ be a separable normed Riesz space. Then the following conditions are mutually equivalent.*

(i) *L_ϱ^a is an order dense ideal in L_ϱ.*

(ii) *${}^0(L_{\varrho,n}^\sim)=\{0\}$, i.e., for every $0<u \in L_\varrho$ there exists an element $0<\varphi \in L_{\varrho,n}^\sim$ such that $\varphi(u)>0$.*

(iii) *There exists on L_ϱ a strictly positive norm bounded normal integral.*

(iv) *L_ϱ has the Egoroff property.*

Proof. (i) $\Rightarrow$ (ii) Given $0<u \in L_\varrho$, there exists $v \in L_\varrho^a$ such that $0<v \leqslant u$. By Corollary 19.4 in Note VI there exists $0<\psi \in L_\varrho^*$ with $\psi(v)>0$. Let ψ_0 be the restriction of ψ on L_ϱ^a. Obviously, ψ_0 is a normal integral on L_ϱ^a. Since ψ_0 has a positive extension to L_ϱ (namely, ψ), ψ_0 has also a normal positive extension φ, and $\varphi(u) \geqslant \varphi(v)=\psi_0(v)=\psi(v)>0$, so φ is the desired element of $L_{\varrho,n}^\sim$.

(ii) $\Rightarrow$ (iii) Since L_ϱ is separable, it follows from Theorem 42.3 in Note XIII that there exists $\varphi \in L_\varrho^*$ such that φ is strictly positive on L_ϱ. Let φ_c be the integral component of φ. Every integral on L_ϱ is normal (cf. again Theorem 42.3 in Note XIII), so φ_c is a normal integral. It follows from $0 \leqslant \varphi_c \leqslant \varphi$ that $\varphi_c \in L_{\varrho,n}^*$, and so the proof will be complete if we show that φ_c is strictly positive, i.e., if the null ideal N_{φ_c} of φ_c satisfies $N_{\varphi_c}=\{0\}$. Assume that $N_{\varphi_c} \neq \{0\}$. It follows then from (ii) and from Lemma 19.5 in Note VI that there exists $0<\psi \in L_{\varrho,n}^\sim$ such that $C_\psi \subset N_{\varphi_c}$, i.e., $\psi \perp \varphi_c$. Now observe that $\psi_0=\inf(\varphi, \psi) \neq 0$. Indeed, if $\inf(\varphi, \psi)=0$, i.e. if $\varphi \perp \psi$, then $C_\varphi \subset N_\psi$ by Theorem 31.2 (i) in Note IX, so $N_\psi=L_\varrho$ or, equivalently, $\psi=0$, which contradicts the definition of ψ. Hence $0<\psi_0 \in L_{\varrho,c}^*$. But then $\varphi_c<\varphi_c+\psi_0 \leqslant \varphi$ with $\varphi_c+\psi_0 \in L_{\varrho,c}^*$, where $\varphi_c+\psi_0 \leqslant \varphi$ follows from $\varphi_c \leqslant \varphi$, $\psi_0 \leqslant \varphi$ and $\varphi_c \perp \psi_0$. This, however, contradicts the definition of φ_c. The final result is that $N_{\varphi_c}=\{0\}$.

(iii) $\Rightarrow$ (iv) Follows from Theorem 31.11 in Note X.

(iv) $\Rightarrow$ (i) Let S^* be the norm closed unit ball in L_ϱ^*, i.e., $S^*=\{\varphi: \varrho^*(\varphi) \leqslant 1\}$. Then, as was observed also in section 43 of Note XIII, S^* is a separable compact metric space in the $\sigma(S^*, L_\varrho)$ topology. Let $\Sigma^*=S^* \cap L_{\varrho,s}^*$, i.e., Σ^* consists of all singular elements in S^*. It follows that Σ^* is a separable metric space in the $\sigma(\Sigma^*, L_\varrho)$ topology. Let $\{\varphi_n: n=1, 2, \ldots; \varphi_n \in \Sigma^*\}$ be dense in Σ^* with respect to this topology. We assert that

$$\bigcap_{n=1}^\infty N_{\varphi_n}=\cap(N_\varphi: \varphi \in \Sigma^*)=\cap(N_\varphi: \varphi \in L_{\varrho,s}^*)=L_\varrho^a.$$

The equality in the middle is trivial, and the last equality is a reformulation of ${}^{\perp}(L^*_{\varrho,s}) = L^a_\varrho$. It remains to prove the first equality, i.e., we have to prove that $\cap N_{\varphi_n} \subset \cap (N_\varphi : \varphi \in \Sigma^*)$. Let $0 < u \in N'_n$ for all n, let $\varphi \in \Sigma^*$ and $\varepsilon > 0$. Now, $\varphi \in \Sigma^*$ implies $|\varphi| \in \Sigma^*$, and since $\{\varphi_n : n = 1, 2, \ldots\}$ is $\sigma(\Sigma^*, L_\varrho)$ dense, there exists n such that $||\varphi|(u) - \varphi_n(u)| < \varepsilon$. Hence $|\varphi|(u) < \varepsilon$. This holds for every $\varepsilon > 0$, so $|\varphi|(u) = 0$, i.e., $u \in N_\varphi$. The desired result follows.

Now, at last, the Egoroff property of L_ϱ will be used. In view of this property every N_{φ_n} is super order dense by Corollary 20.7 in Note VI, and so $L^a_\varrho = \bigcap_{n=1}^{\infty} N_{\varphi_n}$ is order dense by the preceding Lemma 44.1.

Corollary 44.3. *If the separable space L_ϱ satisfies one (and hence each) of the conditions in the preceding theorem, then the conditions in Theorem* 24.6 (*in* Note VII) *are also satisfied, i.e., $(L^a_\varrho)^\perp = L^*_{\varrho,s}$ or, equivalently, $(L^a_\varrho)^* = L^*_{\varrho,c}$ holds.*

Proof. Since $(L^a_\varrho)^\perp \supset L^*_{\varrho,s}$ is always satisfied, it is sufficient to show that any $0 \leqslant \varphi \in L^*_{\varrho,c}$ which vanishes on L^a_ϱ vanishes identically. This is evident since L^a_ϱ is super order dense.

Note that the space L_ϱ of all real continuous functions on $\{x : 0 \leqslant x \leqslant 1\}$, with ϱ the uniform norm, is separable, and $(L^a_\varrho)^\perp = L^*_{\varrho,s}$ is satisfied on account of $L^a_\varrho = \{0\}$ and $L^*_{\varrho,s} = L^*_\varrho$, but L^a_ϱ is not order dense in L_ϱ. The same happens in Example 43.5, where the space L_ϱ is even super Dedekind complete.

The question may be raised if every normed Riesz space for which the conditions (i)–(iv) of Theorem 44.2 are all satisfied is necessarily separable. The answer is negative; the space l_∞ is a counterexample.

We conclude the section with the following example in order to show that Theorem 44.2 is best possible in the sense that, even in the case that L_ϱ is separable and super Dedekind complete, we cannot replace condition (i) by the condition that $L_\varrho = L^a_\varrho$.

Example 44.4. Let L be the Riesz space l_∞ of all real bounded sequences and let φ be a linear functional on L which has the additional properties (a) $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in L$, i.e., φ is multiplicative, (b) $\varphi(f) = 0$ whenever $f \in c_0$ and (c) there is an element $f \in L$ such that $\varphi(f) \neq 0$. In order to prove the existence of a linear functional with properties (a), (b) and (c) observe that $L = l_\infty$ is an algebra with a unit in which c_0 is a proper ideal. Let I be any proper maximal ideal containing c_0. Then the homomorphism φ of $L = l_\infty$ onto $l_\infty / I = R$ is a linear functional satisfying (a), (b) and (c). In addition to (a), (b) and (c) such a linear functional has also the following properties. (d) $\varphi(1) = 1$ and $f \geqslant 0$ implies $\varphi(f) \geqslant 0$. Indeed, $\varphi(f^2) = (\varphi(f))^2$ implies that φ is positive and that $\varphi(1) = 1$. (e) $\varphi(|f|) = |\varphi(f)|$ for all $f \in L$. Indeed, $\varphi(|f|) = \sqrt{\varphi(f^2)} = |\varphi(f)|$. Thus φ is a lattice homomorphism as well. In particular, $\{f : \varphi(f) = 0\} = \{f : \varphi(|f|) = 0\} = N_\varphi$, or in other words, the null space of φ is also the null ideal of φ. (f) φ is singular. Indeed, $\varphi=\varphi_c+\varphi_s$ implies that $N_{\varphi_c}\supset N_\varphi$, and in particular that φ_c vanishes on c_0 and hence vanishes everywhere on l_∞.

Consider now the following norm on $L=l_\infty$: $\varrho(f)=\sum_1^\infty|f(n)|/2^n+\varphi(|f|)(f\in L)$. Furthermore, consider the measure μ on N, the set of natural numbers, with the property that $\mu(\{n\})=2^{-n}(n\in N)$ and let L_1 be the space of all real f on N such that $\varrho_1(f)=\sum_1^\infty|f(n)|/2^n<\infty$. Then L_1 is a separable normed Riesz space. Hence, the normed Riesz space $L_1\oplus R$, where R has the usual Euclidean norm, is also a separable normed Riesz space. The mapping $f\to f\oplus\varphi(f)$ of L_ϱ into $L_1\oplus R$ imbeds L_ϱ in a $1-1$ fashion into a Riesz subspace of $L_1\oplus R$. This imbedding also preserves the norm. Indeed, $\varrho(f)=\sum_1^\infty|f(n)|/2^n+\varphi(|f|)=\sum_1^\infty|f(n)|/2^n+|\varphi(f)|$, which is the norm of the element $f\oplus\varphi(f)$ in $L_1\oplus R$. Hence, L_ϱ is separable, and since l_∞ is super Dedekind complete and has the Egoroff property L_ϱ is a separable super Dedekind complete normed Riesz space having the Egoroff property. But $L_\varrho^a={}^\perp(L_{\varrho,s}^*)\subset N_\varphi=I$ and $N_\varphi\subset L_\varrho^a$ shows that $L_\varrho^a=N_\varphi\neq L_\varrho$, i.e., L_ϱ^a is a proper order dense ideal containing c_0.

45. *Separability and reflexivity*

In this section it will be shown how the results obtained in section 40 of the preceding Note XIII for reflexivity of a normed Riesz space L_ϱ can be applied in the particular case that L_ϱ is separable in the norm topology. We recall that if E is a normed linear space and the first Banach dual E^* is separable, then E itself is separable.

Theorem 45.1. *If L_ϱ^{**} is separable, then L_ϱ^* (and hence also L_ϱ^{**}) is reflexive. If, in addition, L_ϱ is norm complete, then L_ϱ is reflexive.*

Proof. We have to prove that L_ϱ^* satisfies the conditions (ii) of Theorem 40.1 of Note XIII (the reflexivity theorem), i.e., we have to prove that $L_\varrho^*=(L_\varrho^*)^a$, $L_\varrho^{**}=(L_\varrho^{**})^a$ and that $0\leqslant\varphi_n\uparrow$ in L_ϱ^* with sup $\varrho^*(\varphi_n)<\infty$ implies the existence of sup φ_n. The proof for the last condition is easy and has nothing to do with separability. Given $0\leqslant\varphi_n$ with sup $\varrho^*(\varphi_n)<\infty$, the number $\varphi_0(u)=\sup\varphi_n(u)$ is finite and non-negative for every $u\in L_\varrho^*$, so (upon extending φ_0 in the obvious way to the whole of L_ϱ) we have $|\varphi_0(f)|\leqslant\varrho(f)$. sup $\varrho^*(\varphi_n)$, i.e., $0\leqslant\varphi_0\in L_\varrho^*$ and sup $\varphi_n=\varphi_0$.

Furthermore, since separability of L_ϱ^{**} implies separability of L_ϱ^*, we have now that L_ϱ^* and L_ϱ^{**} are separable, norm complete and super Dedekind complete, so $L_\varrho^*=(L_\varrho^*)^a$ and $L_\varrho^{**}=(L_\varrho^{**})^a$ by Theorem 42.8 in Note XIII.

The above theorem is due to T. Ogasawara ([8], Ch. V, § 4, Theorem 3). It is well-known that the theorem is not true for an arbitrary Banach space; there exists a nonreflexive Banach space B such that B^{**} is separable (cf. R. C. JAMES [3]).

Corollary 45.2. *If L_ϱ^{**} is separable, then L_ϱ has the property* (A, iii), *i.e., every monotone order bounded sequence is a ϱ-Cauchy sequence.*

Proof. From the preceding theorem it follows that ϱ^{**} has (A, ii) and hence ϱ has at least (A, iii).

Example 45.3. Let $L=C(X)$ be the Riesz space considered in Example 43.5. Consider now the following norm on L: $\varrho(f)=\sqrt{(\sum_1^\infty |f(x_n)|^2/2^n)}$, where $\{x_n\}$ is a countable dense subset of X. Then in the same way as in Example 43.5 it follows that L_ϱ is separable and super Dedekind complete. Furthermore, L_ϱ being a separable inner product space its first Banach dual L_ϱ^* is separable and reflexive. But $L_\varrho^a=\{0\}$.

Theorem 45.4. *If L_ϱ has the sequential weak Fatou property (i.e., if $0\leqslant u_n\uparrow$ in L_ϱ with sup $\varrho(u_n)<\infty$, then sup u_n exists) and if L_ϱ^* is separable, then L_ϱ is reflexive.*

Proof. The sequential weak Fatou property implies σ-Dedekind completeness and norm completeness, so L_ϱ and L_ϱ^* are separable, norm complete and σ-Dedekind complete. It follows then from Theorem 42.8 in Note XIII that $L_\varrho=L_\varrho^a$ and $L_\varrho^*=(L_\varrho^*)^a$. The space L_ϱ satisfies therefore the conditions (ii) of Theorem 40.1 in Note XIII (the reflexivity theorem), so L_ϱ is reflexive.

The normed Riesz space c_0 of all null sequences with the supremum norm furnishes an example of a nonreflexive normed Riesz space which is norm complete, super Dedekind complete and whose first Banach dual l_1 is separable.

The normed Riesz space c of all convergent sequences with the supremum norm on the other hand shows that there exists a normed Riesz space which is norm complete, whose first Banach dual space ($l_1\oplus R$) is separable, but which does not have the property (A, iii). This shows that in Corollary 45.2 we cannot replace the hypothesis that L_ϱ^{**} is separable by the weaker hypothesis that L_ϱ^* is separable. Nevertheless, normed Riesz spaces whose first Banach duals are separable do have some interesting additional properties which we shall discuss below.

If L_ϱ^* is separable, then Theorem 42.1 in Note XIII can be strengthened as follows.

Theorem 45.5. *If L_ϱ is a normed Riesz space such that L_ϱ^* is separable, then there exists a countable order basis in L_ϱ which is also an order basis for L_ϱ^{**}. If L_ϱ is norm complete and L_ϱ^* is separable, then there exists an element $0<u_0\in L_\varrho$ which is a weak unit for L_ϱ^{**}.*

Proof. From Theorem 42.1 in Note XIII it follows that there exists a countable order basis $\{u_n\}$ of positive elements in L_ϱ which at the same time is norm dense in the set of non-negative elements in L_ϱ. In order to prove that $\{u_n\}$ is order dense in L_ϱ^{**} we have to show that $0 \leqslant u'' \in L_\varrho^{**}$ and $\inf(u'', u_n) = 0$ for all n implies $u'' = 0$. From the hypothesis that L_ϱ^* is separable, norm complete and Dedekind complete, and from Theorem 42.8 in Note XIII it follows that $L_\varrho^{**} = (L_\varrho^*)_n^*$. Then $\inf(u'', u_n) = 0$ implies by Theorem 27.6 in Note VIII that $C_{u''} \subset N_{u_n}$, where $C_{u''}$ is the carrier of the normal integral u'' on L_ϱ^* and N_{u_n} is the null ideal of the normal integral u_n on L_ϱ^*. Since the set $\{u_n\}$ is norm dense in the positive cone of L_ϱ it follows that $\cap N_{u_n} = \{0\}$. So $C_{u''} = \{0\}$, i.e., $u'' = 0$.

The second part of the theorem is now evident.

The following theorem which is of interest in itself is of importance for the next result.

Theorem 45.6. *Let L_ϱ be a normed Riesz space and let $H \subset L_\varrho$ be a subset of L_ϱ which has the property that every sequence of elements of H has a subsequence which is a weak Cauchy sequence. Then the Riesz seminorm $\varrho_H(\varphi) = sup\ (|\varphi|(|f|) : f \in H)$ on L_ϱ^* has the property* (A, iii), *i.e., every monotone orderbounded sequence in L_ϱ^* is a ϱ_H-Cauchy sequence. In particular, if H is weakly relatively compact, then ϱ_H has the property* (A, iii).

Proof. From the principle of uniform boundedness it follows immediately that H is norm bounded, and hence $\varrho_H(\varphi) < \infty$ for all $\varphi \in L_\varrho^*$. If ϱ_H does not have the property (A, iii), then there exists a sequence $0 \leqslant \varphi_n \uparrow \leqslant \varphi_0$ and a positive number $\varepsilon > 0$ such that $\varrho_H(\varphi_{n+1} - \varphi_n) > \varepsilon$. Hence, there exist elements $f_n \in H$ $(n = 1, 2, \ldots)$ such that $(\varphi_{n+1} - \varphi_n)(|f_n|) > \varepsilon$. From the hypothesis on H it follows that we may assume that $\{f_n\}$ is a weak Cauchy sequence. Since

$$(\varphi_{n+1} - \varphi_n)(|f_n|) = \sup\ (\psi(f_n) : |\psi| \leqslant \varphi_{n+1} - \varphi_n)$$

it follows that for every n there is an element ψ_n such that $|\psi_n| \leqslant \varphi_{n+1} - \varphi_n$ and $\psi_n(f_n) > \varepsilon$. For every subset $A \subset N$, the set of natural numbers, we set

$$\psi_A = \sup\ (\psi_F : F \subset A \text{ and } F \text{ is finite}),$$

where $\psi_F = \sum_{n \in F} \psi_n$. Since $|\psi_F| \leqslant \varphi_0$ it follows that ψ_A exists for every $A \subset N$ and that $|\psi_A| \leqslant \varphi_0$. If A, $B \subset N$ and $A \cap B = \emptyset$, then $\psi_A + \psi_B = \psi_{A \cup B}$. Indeed, if $F \subset A \cup B$, F is finite, then $\psi_F = \psi_{F \cap A} + \psi_{F \cap B}$, and hence $\psi_{A \cup B} \leqslant \psi_A + \psi_B$; if $F_1 \subset A$ and $F_2 \subset B$, F_1, F_2 finite, then $\psi_{F_1} + \psi_{F_2} = \psi_{F_1 \cup F_2} \leqslant \psi_{A \cup B}$ implies $\psi_A + \psi_B \leqslant \psi_{A \cup B}$. Now we set $\nu_n(A) = \psi_A(f_n)$ for every $A \subset N$. Then A, $B \subset N$ and $A \cap B = \emptyset$ implies that $\nu_n(A \cup B) = \nu_n(A) + \nu_n(B)$, and $|\nu_n(A)| \leqslant \varphi_0(|f_n|)$ for all $A \subset N$ shows that ν_n is a finitely additive signed measure on the Boolean algebra of all subsets of N. Since $\{f_n\}$ is a weak Cauchy sequence we have that $\lim \nu_n(A)$ exists for every

$A \subset N$. But then as in the proof of Theorem 43.1 it follows from a lemma of R. S. PHILLIPS ([9], Lemma 3.3) that $\lim \nu_n(\{n\}) = 0$, contradicting $\psi_n(f_n) > \varepsilon$ for all n.

If $H \subset L_\varrho$ is weakly relatively compact, then it follows from a well-known result of V. ŠMULIAN [10] that every sequence of H contains a converging subsequence, and so H satisfies the hypothesis of the theorem (for a proof, cf. N. DUNFORD and J. T. SCHWARTZ, Linear operators *I*, Ch. V, § 6, Theorem 1).

The following theorem parallels Theorem 43.3.

Theorem 45.7. *Let L_ϱ be a normed Riesz space such that L_ϱ^* is separable. Then there exists a strictly positive linear functional $0 < \varphi \in L_\varrho^*$ on L_ϱ such that*

(i) *$u_\tau \downarrow$ and $\varphi(u_\tau) \downarrow 0$ implies $\varrho(u_\tau) \downarrow 0$.*

(ii) *$0 \leqslant u_n \leqslant u$ ($n = 1, 2, \ldots$) and $\lim \varphi(u_n) = 0$ implies that $\{u_n\}$ converges to zero uniformly on every weak Cauchy sequence in L_ϱ^*. In particular, if $0 \leqslant u_n \leqslant u$ and $\{u_n\}$ converges weakly to zero, then $\{u_n\}$ converges to zero uniformly on every weakly relatively compact subset of L_ϱ^*.*

Proof. (i) Since L_ϱ^* is separable and norm complete it follows from Theorem 42.2 in Note XIII that there is an element $0 < \varphi \in L_\varrho^*$ which is strictly positive on L_ϱ^{**}. Let $0 \leqslant u_\tau \downarrow$ and $\varphi(u_\tau) \downarrow 0$. Then $u_\tau \downarrow 0$ in L_ϱ^{**}, and hence $L_\varrho^* \subset (L_\varrho^{**})_n^*$ implies that $\psi(u_\tau) \downarrow 0$ for every $\psi \in L_\varrho^*$, so $u_\tau \downarrow 0$ weakly. But then by Mazur's theorem as in the proof of Lemma 22.6 in Note VII it follows that $\varrho(u_\tau) \downarrow 0$.

(ii) Let $0 \leqslant u_n \leqslant u$ ($n = 1, 2, \ldots$), $\lim \varphi(u_n) = 0$ and let $H \subset L_\varrho^*$ satisfy the hypothesis of Theorem 45.6. Then the seminorm $\varrho_H(f) = \sup (|\varphi|(|f|): \varphi \in L_\varrho^*)$, $f \in L_\varrho$, has the property (A, iii) and $\varrho_H \leqslant \sup (\varrho^*(\varphi): \varphi \in H)\varrho$. Hence, from the proof of part (ii) of Theorem 43.3 it follows then that $\varrho_H(u_n) \to 0$ as $n \to \infty$.

(To be continued)